linear quadratic controller design theory. The approach differs from foregoing work in three points: the use of a scalar state formulation which is valid near impact, the use of an optimization interval for which the terminal time is an open design parameter, and the use of the standard regulator format with running cost on the system state. The resulting kinematic guidance law is more general than previous results, since it produces the whole range of classical proportional navigation ratios on the one hand, and, on the other hand, it shows that the proportional navigation law (up to interception) is just an extreme case of guidance to handover to a final blind period. Final blind periods are significant in practice due to seeker measurement corruptions and image explosions near impact. Thus the practical problem is often to minimize a performance criterion continuously during flight and at the final blind time, and not exclusively at impact—a fact which in most theoretical assessments is not considered.

References

[1] Murtaugh, S.A. and Criel, H.E., "Fundamentals of Proportional Navigation," *IEEE Spectrum*, Vol. 3, Dec. 1966, pp. 75-85.

[2] Bennett, R.R. and Mathews, W.E., "Analytical Determination of Miss Distances for Linear Homing Navigation Systems," Hughes Aircraft Co., Culver City, Calif., TM 260, March 1952.

[3] Bryson, A.E. Jr., "Linear Feedback Solutions for Minimum Effort Interception, Rendezvous and Soft Landing," *AIAA Journal*, Vol. 3, Aug. 1965, pp. 1542-1544.

[4] Garber, V., "Optimum Intercept Laws for Accelerating Targets," *AIAA Journal*, Vol. 6, Nov. 1968, pp. 2196-2198.

[5] Dickson, R. and Garber, V., "Optimum Rendezvous, Intercept, and Injection," *AIAA Journal*, Vol. 7, July 1969, pp. 1402-1403.

[6] Willems, G.C., "Optimal Controllers for Homing Missiles with Two Time Constants," Army Missile Command, Redstone Arsenal, Ala., Rept. RE-TR-60-20 (AD 862 093), Oct. 1969.

[7] Kreindler, E., "Optimality of Proportional Navigation," *AIAA Journal*, Vol. 11, June 1973, pp. 878-880.

[8] Pastrick, H.L., Seltzer, S.M., and Warren, M.E., "Guidance Laws for Short Range Tactical Missiles," *Journal of Guidance and Control*, Vol. 4, March-April 1981, p. 103.

An Analytic Solution for the State Trajectories of a Feedback Control System

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Introduction

As part of the normal process of a control system design, the analyst frequently is interested in determining the state trajectories for the controlled system. In practice, this process is straightforward, since the feedback form of the control can be introduced in the equation of motion and numerically integrated. Nevertheless, this process can be computationally intensive, if either time-varying control gains are used, or if small integration step sizes are required by the presence of high-frequency system dynamics.

In an effort to overcome the computational difficulties listed above, we present in this Note a change of variables for the closed-loop system dynamics equation, which permits a closed-form expression to be obtained for the state trajectories.

Optimal Control Problem

The fixed time linear optimal control problem is formulated by finding the control inputs $u(t)$ to minimize

$$J=\frac{1}{2}x_f^T F^T S F x_f+\frac{1}{2}\int_{t_0}^{t_f}(x^T F^T Q F x+u^T R u)\,\mathrm{d}t \tag{1}$$

for the system

$$\dot{x}=Ax+Bu, \quad \text{given} \quad x(t_0) \tag{2}$$

$$y=Fx \tag{3}$$

where x is the state, u is the control, A is the system dynamics matrix, B is the control influence matrix, F is the measurement influence matrix, Q is the output weighting matrix, R is the control weight matrix, and S is the terminal output weight matrix.

As shown in Ref. 1, the optimal control is given by

$$u(t)=-R^{-1}B^T P(t)x(t) \tag{4}$$

where P is the solution to the differential matrix Riccati equation

$$\dot{P}=-PA-A^T P+PBR^{-1}B^T P-F^T QF; \quad P(t_f)=F^T SF \tag{5}$$

Upon introducing Eq. (4) into Eq. (2), the standard closed-loop system dynamics equation follows as

$$\dot{x}(t)=[A-BR^{-1}B^T P(t)]x(t); \quad x_0=x(t_0) \tag{6}$$

To simplify the solution to Eq. (6), we introduce the following closed-form solution for $P(t)$[2-8] in Eq. (5):

$$P(t)=P_{ss}+Z^{-1}(t) \tag{7}$$

where P_{ss} is the solution to the algebraic Riccati equation[9,10]

$$-PA-A^T P+PBR^{-1}B^T P-F^T QF=0 \tag{8}$$

and $Z(t)$ is a matrix function which is to be determined.

Upon introducing Eq. (7) and its time derivative into Eq. (5), the linear constant coefficient matrix differential equation for $Z(t)$ follows as

$$\dot{Z}=\bar{A}Z+Z\bar{A}^T-BR^{-1}B^T; \quad Z(t_f)=(F^T SF-P_{ss})^{-1} \tag{9}$$

from which it follows that the solution for $Z(t)$[11,12] is given by

$$Z(t)=Z_{ss}+e^{\bar{A}(t-t_f)}[Z(t_f)-Z_{ss}]e^{\bar{A}^T(t-t_f)} \tag{10}$$

where $\bar{A}=A-BR^{-1}B^T P_{ss}$ is the steady-state closed-loop system matrix, $e^{(\cdot)}$ is the exponential matrix, and Z_{ss} satisfies the algebraic Lyapunov equation[10,13-15]

$$\bar{A}Z_{ss}+Z_{ss}\bar{A}^T=BR^{-1}B^T$$

Substituting Eq. (7) into Eq. (6) yields the modified form of the closed-loop system dynamics equation

$$\dot{x}(t)=[\bar{A}-BR^{-1}B^T Z^{-1}(t)]x(t); \quad x_0=x(t_0) \tag{11}$$

where we observe that the equation above is nonautonomous.

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Change of Variables

To simplify Eq. (11), we introduce the following coordinate transformation for the dependent variable $x(t)$:

$$x(t)=Z(t)r(t) \tag{12}$$

where $Z(t)$ is given by Eq. (10) and $r(t)$ is a vector function which is to be determined.

Upon differentiating Eq. (12) we find

$$\dot{x}=\dot{Z}r+Z\dot{r} \tag{13}$$

or

$$\dot{x}=[\bar{A}Z+Z\bar{A}^T-BR^{-1}B^T]r+Z\dot{r} \tag{14}$$

where $\dot{Z}$ in Eq. (13) has been replaced by the right side of Eq. (9).

The differential equation for $r(t)$ is obtained by introducing Eqs. (12) and (14) into Eq. (11), leading to

$$Z(\dot{r}+\bar{A}^Tr)=0 \tag{15}$$

from which it follows that the linear constant coefficient vector differential equation for $r(t)$ is given by

$$\dot{r}=-\bar{A}^Tr; \quad r_0=Z^{-1}(t_0)x_0 \tag{16}$$

where the solution for $r(t)$ follows as

$$r(t)=e^{-\bar{A}^T(t-t_0)}r_0 \tag{17}$$

Substituting Eq. (17) into Eq. (12) produces the desired solution for the state trajectories as

$$x(t)=\Phi(t,t_0)x_0 \tag{18}$$

where $\Phi(t,t_0)=Z(t)e^{-\bar{A}^T(t-t_0)}Z^{-1}(t_0)$ is the system state transition matrix, and $\Phi(t,t_0)$ satisfies the following matrix differential equation

$$\dot{\Phi}(t,t_0)=[A-BR^{-1}B^TP(t)]\Phi(t,t_0); \quad \Phi(t_0,t_0)=I$$

Recursion Relationship for Evaluating the State at Discrete Times

If the solution for $x(t)$ is required at the discrete times $t_k=t_0+k\Delta t$ $(k=1,...,N)$ for $\Delta t=(t_f-t_0)/N$, then Eq. (18) can be written as

$$x(t_k)=Z_{ss}a_k+b_k \qquad (k=0,...,N) \tag{19}$$

where

$$a_0=r_0$$

$$b_0=e^{\bar{A}(t_0-t_f)}[Z(t_f)-Z_{ss}]e^{\bar{A}^T(t_0-t_f)}r_0$$

$$a_k=e^{-\bar{A}^T\Delta t}a_{k-1}$$

$$b_k=e^{\bar{A}\Delta t}b_{k-1}$$

Conclusions

A straightforward algorithm has been presented for generating the state trajectories for a feedback control system. The algorithm is computationally efficient in that no numerical integration is required and simple recursion relationships generate the desired solution at discrete times. Furthermore, this algorithm has signficant potential if used in conjunction with algorithms which attempt to enhance system robustness by iteratively refining the weighting matrices appearing in the performance index.

References

[1] Bryson, A., and Ho, Y.C., *Applied Optimal Control,* John Wiley and Sons, New York, 1975.

[2] Turner, J.D., and Chun, H.M., "Optimal Feedback Control of a Flexible Spacecraft During a Large-Angle Rotational Maneuver," AIAA Paper No. 82-1589-CP, 1982.

[3] Potter, J.E., and Vander Velde, W.E., "Optimal Mixing of Gyroscope and Star Tracker Data," *Journal of Spacecraft,* Vol. 5, May 1968, pp. 536-540.

[4] Brockett, R.W., *Finite Dimensional Linear Systems,* John Wiley and Sons, Inc., New York, 1970.

[5] Turner, J.D., Chun, H.M., Juang, J.N., "Optimal Slewing Maneuvers for Flexible Spacecraft Using a Closed Form Solution for the Linear Tracking Problem," AIAA Paper No. 83-374, AAS/AIAA Astrodynamics Conference, Aug. 22-25, 1983, Lake Placid, N.Y.

[6] Potter, J.E., "A Matrix Equation Arising in Statistical Filter Theory," NASA CR-270, 1965.

[7] Martensson, K., "On the Matrix Riccati Equation," *Information Sciences,* Vol. 3, 1971, pp. 17-49.

[8] Prussing, J.E., "A Simplified Method for Solving the Matrix Riccati Equation," *International Journal of Control,* Vol. 15, No. 5, 1972, pp. 995-1000.

[9] Potter, J.E., "Matrix Quadratic Solutions," *SIAM Journal of Applied Mathematics,* Vol. 14, No. 3, 1964, pp. 496-501.

[10] Jamshidi, M., "An Overview on the Solutions of the Algebraic Matrix Riccati Equation and Related Problems," *Large Scale Systems: Theory and Application,* M.G. Singh and A.P. Sage, eds., Elsevier North-Holland Pub. Co., Vol. 1, No, 3, Aug. 1980, pp. 167-192.

[11] Davison, E.J., "The Numerical Solution of $\dot{X}=A_1X+XA_2+D$, $X(0)=C$, *IEEE Transactions on Automatic Control,* Vol. AC-20, Aug. 1975, pp. 566-567.

[12] Serbin, S.M. and Serbin, C.A., "A Time-Stepping Procedure for $\dot{X}=A_1X+XA_2+D$, $X(0)=C$, *IEEE Transactions on Automatic Control,* Vol. AC-25, No. 6, Dec. 1980, pp. 1138-1141.

[14] Bartels, R.H. and Stewart, G.W., "A Solution of the Equation $AX+XB=C$," *Communications of the ACM,* Vol. 15, No. 9, Sept. 1972, pp. 820-826.

[15] Golub, G.H., Nash, S., and Van Loan, C., "A Hessenburg-Schur Method for the Problem $AX+XB=C$," *IEEE Transactions on Automatic Control,* Vol. AC-24, No. 6, Dec. 1979, pp. 909-913.

Isochrones for Maximum Endurance Horizontal Gliding Flight

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Nomenclature

C_1,C_2,C_3	= constants of integration
C_D	= drag coefficient, $C_{D0}+KC_L^2$
C_{D0}	= zero lift drag coefficient, constant
C_L	= lift coefficient
C_L^*	= C_L for maximum lift-to-drag ratio
D	= drag
E^*	= maximum lift-to-drag ratio
g	= gravitational acceleration
H	= Hamiltonian function
K	= induced drag factor, constant
L	= lift
n	= load factor, $L/W=1/\cos\mu$
p_x,p_y,p_u,p_ψ	= adjoint variables

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